Creating Multiple Representations in Algebra: ALL CHOCOLATE,

ARTHUR HYDE, KATIE GEORGE, SUZANNE MYNARD, CHRISTINA HULL, SHARON WATSON, AND PATRICK WATSON

AS TEACHERS OF MATHEMATICS IN THE middle grades, we were excited to find an approach to teaching algebra concepts that could be readily modified to be successful with all our students. *Principles and Standards for School Mathematics* (NCTM 2000) is explicit about the importance of students learning algebra to represent quantitative relationships: "Students should solve problems in which they use tables, graphs, words, and symbolic expressions to represent and examine functions and patterns of change" (p. 223).

If we are to make algebra accessible to more students, especially those who are traditionally underserved, we need approaches and problem situations that entice students to attack them mathematically. For most students, the initial development of meaning for algebraic concepts is highly context-dependent. Students build conceptual understanding through extensive work with multiple representations of a particular concept in one context (Lesh, Lester, and Hjalmarson 2003, pp. 397–401). In general, they should move from relatively concrete representations to increasingly more abstract representations of that same concept. Our experience has been that the traditional approach of starting with an equation without a context (the most abstract representation), then making a table of values and later a graph all have little meaning for most of our students.

In the past six years, we have used problems in contexts of high interest to students and have carefully helped them create representations of the situations. Powerful understanding occurs when they identify connections between representations. The context we will describe in detail starts with the question we pose to the students:

> If you have \$10 to spend on \$2 Hershey's bars and \$1 Tootsie Rolls, how many ways can you spend all your money without receiving change? All chocolate, no change.

The problem seems quite simple and is accessible to a wide range of students. However, we can readily increase the level of difficulty by changing the dollar amounts. We have used this approach with

ARTHUR HYDE, *ahyde5@ameritech.net, is a professor of mathematics education at National-Louis University in Lisle, Illinois. His major interest is helping elementary and middle school teachers inspire their students to know the beauty and power of mathematics.* KATIE GEORGE, *bkgeorge3@comcast.net, teaches prealgebra and algebra to seventh graders at Daniel Wright Junior High School in Lincolnshire, Illinois. She wants to make algebra at the middle school level more accessible to all students through innovative approaches.* SUZANNE MYNARD, *szmynard@comcast.net, teaches algebra to seventh and eighth graders at Algonquin Middle School in Des Plaines, Illinois. She researches finding unique and engaging ways to teach mathematics.* CHRISTINA HULL, *chullz@ameritech.net, teaches third-, fourth-, and fifth-grade gifted students at Madison Elementary School in Wheaton, Illinois. She wants to learn ways to help children build deep conceptual understanding before learning a procedure.* SHARON WATSON, *watson.patrick@sbcglobal.net, teaches prealgebra and algebra to seventh graders at West Chicago Middle School in West Chicago, Illinois. She is interested in ensuring that mathematics makes sense to students as a set of tools they can access and use in real-world situations.* PATRICK WATSON, *watson.patrick@sbcglobal.net, teaches mathematics to gifted fifth graders in West Chicago, Illinois. He works to help children make connections between mathematics and literature and contexts in their everyday lives.*

NO CHANGE

students in grades 5–8 in regular mathematics classes and in both prealgebra and algebra classes in several ways: to build a strong base for representing real-world situations with tables and graphs, to introduce linear equations, and to bridge linear equations and systems of equations. We have spent from two to five class periods on this approach because of the rich connections, powerful understandings, and virtually unlimited number of problems that can be created as extensions.

First, our students discuss the situation to make sure they understand the problem, then they create data tables of the possible solutions. Next, we ask questions to help them analyze the patterns in the tables more thoroughly than is traditionally done in prealgebra or algebra classes. Students create graphs, analyze their patterns, and connect them to the patterns in the tables. We help students link their understanding of these patterns back to the real-world situation to give contextual meaning. Our more experienced students create equations from the patterns in the tables. We repeat this process with more difficult problems to build their generalized understanding of key concepts.

This approach uses deceptively simple situations that can be represented by linear equations in the form $ax + by = c$. Technically, these are Diophantine linear equations, because we have ensured that only positive integers are allowed for a, b, c, x, and y (Burton 1991, pp. 234–39; Robbins 1993, p. 72). We have guaranteed first-quadrant graphing and patterns of simple covariation that are relatively easy for students to understand. The following describes an activity with a seventh-grade class, modifications used with fifth and sixth graders, and some extensions done with our more experienced eighth graders.

Presenting the Problem Situation

ARMED WITH A GIANT BAR OF CHOCOLATE AND A king-sized box of Tootsie Rolls, I prepared my initial attack on linear equations with my seventh-grade prealgebra students. Bringing in large candy bars proved to be a motivating factor for the students. From my experience doing this activity several times, I was keenly aware that “chocolate algebra” hinges on moving at the right pace and asking probing questions that drive the students’ conceptual development. I prepared quite a few questions in advance and began by posing the problem to my class, “If you have \$10 to spend on \$2 Hershey’s bars and \$1 Tootsie Rolls, how many ways can you spend all your money without receiving change? All chocolate, no change.” The students quickly began generating solutions. As expected, most jotted down random combinations. As they shared solutions, it became apparent to them that they needed an organizational system.

Creating a Data Table

I SUGGESTED THAT EACH STUDENT MAKE A simple two-column table, or T chart, to record the combinations. We decided together to label the left column “Number of \$2 Hershey’s Bars” and label the right column “Number of \$1 Tootsie Rolls” (see **fig. 1**). The combinations initially elicited from the class were not arranged in any particular order, so I asked them, “Did you find all the possible combinations?” and “How do you know when you have found them all?” To make it easier to ensure that all possibilities were listed, we agreed to purchase the most \$2 Hershey’s bars that we could as a place to start (“the most of the bigger item”) and decrease the

Total $10	
NUMBER OF $2 HERSHEY'S BARS	NUMBER OF $1 TOOTSIE ROLLS
5 ($10)	0 ($0)
4 ($8)	2 ($2)
3 ($6)	4 ($4)
2 ($4)	6 ($6)
1 ($2)	8 ($8)
0 ($0)	10 ($10)

Fig. 1 Ways to spend $10 on Hershey's bars and Tootsie Rolls, illustrating the use of "total bubbles"

number of Hershey's bars one at a time. The first row of our table showed that a total of 5 Hershey's bars and 0 Tootsie Rolls could be purchased. The second row contained 4 Hershey's bars and 2 Tootsie Rolls.

I suggested that they use what I call "total bubbles" in the table. I asked the students to place a small circle to the right and near the top of each number in the two columns. See **figure 1**. In this circle, they wrote the cost of buying the number of those items in the table. This step has been extremely helpful to all my students, especially those who had trouble keeping track of how much they were spending. By adding the two bubbles in the row, they could check on their calculations of possible solutions. The strategy can also help students later when they are writing equations.

The students quickly saw patterns in their table. I asked the class to explain to me how the numbers of the two items changed in the table.

"The left side goes down by 1 and the right side goes up by 2," one student exclaimed.

"Why?" I asked.

"Will the left side always go down and the right side always go up?" another student asked.

"Will the numbers at the top always make that pattern?" a different student asked.

"Where does the pattern come from?" I asked.

"Because the Hershey's bar was twice as expensive, there would be a 2:1 relationship," some students volunteered.

"Oh, the Hershey's bar is exchanged for two Tootsie Rolls!" one student blurted out.

"Look at the total bubbles and describe what happens when you buy 1 fewer Hershey's bars," I said.

"I get it! You save $2, but you've got to spend all your money; so you spend the two extra dollars on buying 2 more of the other," one student said.

Total $37	
NUMBER OF $5 TOBLERONES	NUMBER OF $2 HERSHEY'S BARS
7 ($35)	1 ($2)
5 ($25)	6 ($12)
3 ($15)	11 ($22)
1 ($5)	16 ($32)

Fig. 2 Ways to spend $37 on Toblerones and Hershey's bars, illustrating the use of total bubbles

I suggested that we call it the "down by 1, up by 2" pattern.

Next, I asked students to buy $5 Toblerone chocolate bars and $1 Tootsie Rolls with $27, again with no change leftover. They all readily made tables to show the combinations and named the pattern "down by 1, up by 5." To extend their thinking, I asked them next to shop for $5 Toblerones and $2 Hershey's bars with $37. They generated the third table (see **fig. 2**). We discussed why this pattern was "down by 2, up by 5" and why they could not simply decrease the $5 Toblerones by 1 each time. They looked at the pattern in the total bubbles, and some saw that $10 not spent on Toblerones meant that $10 could be spent on Hershey's bars. One said, "With $10, I could get 5 Hershey's Bars or 2 Toblerones; it's like a 5-to-2 ratio." Nearly all the students understood the tables, liked the total bubbles, and recognized the patterns.

Graphing the Data

ON THE SECOND DAY, WE PULLED OUT OUR tables and reviewed the situations from the previous class. Then we took the first table from the initial situation ($10 to spend on $2 Hershey's bars and $1 Tootsie Rolls). I had copied this table on the board and wrote the solution (5, 0) next to the first row of the table. They immediately recognized it as being a coordinate point in a graph. They created a first-quadrant graph of the six solutions in the table. Some noticed that the solution points "lined up" and that they started in the top left and went down toward the bottom right. They commented that this was different from their usual graphs that went up from the lower left to the top right. I asked them, "Why do these points go down from the top left?" Someone noted that in our current table, one side goes down and other side goes up. In our usual graphs, both sides go up.

I asked them if there were any other solution points, and they all said no. I then asked them what it would mean if we drew a solid line connecting all

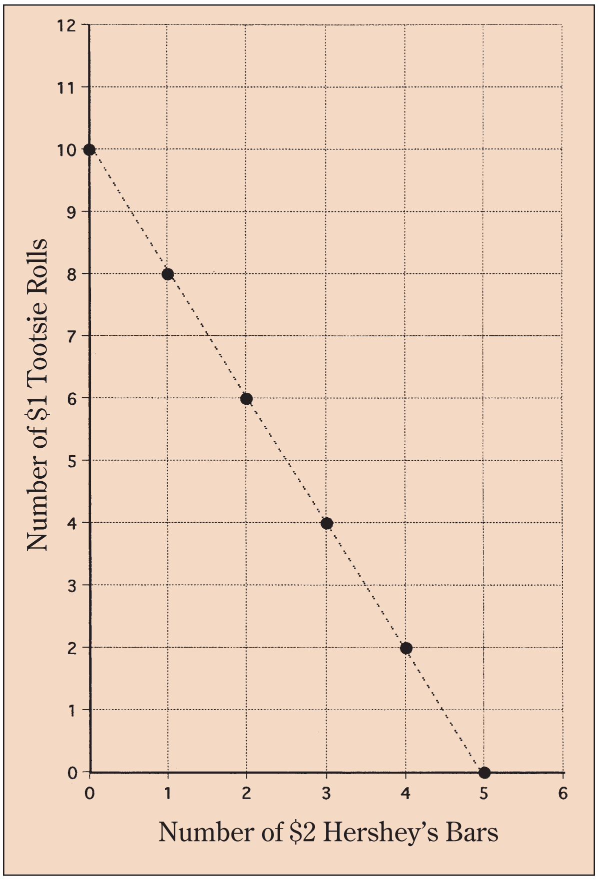

Fig. 3 A graph of all the solutions to the ways to spend $10 on Hershey's bars and Tootsie Rolls

the points. After a while, one student ventured to ask, "Would that mean there were other solutions on that line?" *Exactly.* So students made a dotted line instead (see **fig. 3**). Then I requested, "Take your pencil and put it on the point (5, 0). Let's say we want to go from this point to (4, 2), the next row down in our table, but our pencil has to stay on the lines like in a video game. Can someone tell me how to move my pen so that it will be on the point (4, 2)?" "Move over to the left," someone called out. "How far should we move?" I responded. "Move over left 1, and then go up 2," a student directed. Almost instantaneously, most students realized what was going on. "That's the pattern in the table, down by 1, up by 2! Cool!" The class delighted in seeing the pattern from the table reappear in the graph. We moved to the left 1 and up 2 several times until our pencils were on the point (0, 10). The class had a good initial understanding of the connection between the table and the graph.

Creating an Equation

LOOKING AT THE THREE TABLES THEY HAD completed, I suggested that they could make an equation from this information. "Look at your first table. What numbers are always staying the same?" Without much hesitation, they recognized that the prices of the chocolate ($2/$1) and our budget ($10) always stayed the same. "Look at the table again. What is changing? What is different in every row?" I asked. They recognized that the number that we are buying changes. "What can we do to show that a number is changing or that we do not know what the number will be?" I inquired. "Use a variable," one said; "x," another contributed. I asked, "What do you want x to represent?" They decided that x could be the number of $2 Hershey's bars and that y could be the number of $1 Tootsie Rolls.

I asked, "If you had x as the number of $2 Hershey's bars, how do you know what to put in the total bubble?" Their first thought was "$\$2 \times x$" and then they decided on "$2x$." They repeated the same process for the $1 Tootsie Rolls to put $\$1 \times y$ or just y in that bubble. I asked, "How could we put all the information together into one general sentence?" The students discussed the question for a few minutes, then using their input I wrote this equation on the chalkboard:

(The cost of the $2 items) + (the cost of the $1 items)
= (the total money we have to spend)

I asked them to substitute what we could use to represent parts of this sentence. Several shouted, "$2x + y = 10$." We had our equation! I lost track of how many students said "Cool."

At the end of the lesson, I asked them to think about what we had done and what they had learned. Many mentioned the use of a table and "the most of the bigger." Others noticed that we were using two variables and solving for them in a different way than we had done in the past for one variable. This introduction would require follow-up and extensions. For homework, I asked students to create a problem using a budget and two items to buy. They should make a table and a graph and try to write an equation. They delighted in designing their own problems and created tables with ease. Most were able to make a graph to go along with their table, but the biggest challenge was in making an equation. In subsequent weeks, they had many more opportunities to create equations.

Modifications for Younger and Special Needs Students

WE USUALLY PLAN THE TASK SO THAT YOUNGER or special needs students can work in pairs for these activities. We give them stacks of play money so that they can have a concrete manipulative to use to help them think about the solutions. Some students need

Total $35.00

$1.75 Item		$1.25 Item	
20	$35.00	0	$0
0	$0	28	$35.00

Fig. 4 Ways to spend $35.00 on $1.75 and $1.25 items, illustrating the box-in-the-corner design

to start very concretely. In the first problem—$10 to spend on $2 and $1 chocolate items—we tell them that each $2 bill represents a $2 Hershey's bar and that a $1 bill represents a Tootsie Roll. They can show us a solution by holding up the bills that represent the chocolate items. We tell them that in real life two $1 bills can buy one $2 item but to help us check our answers, use $1 bills only for Tootsie Rolls.

When making data tables, we have found that our students need for us to emphasize that the numbers in the table are referring to *how many* chocolate items of each kind, not the money involved in the purchase. We give the students a recording sheet for the T tables. Many of these students cannot retain as much information in working memory as older students. We use either the total bubbles, as shown in **figures 1** and **2**, or the *box-in-the-corner design* (shown in **fig. 4**). Either design would work, since each gives students a place to record the cost (How much?) of the number of items (How many?) they have entered into rows of the table. We help them see that the two boxes for each row must sum to the total amount of money they have to spend. It also helps them generate the data systematically. For many of our students, this device is a major factor in helping them understand the pattern in the tables. Some of us find it a little easier to create a recording sheet with boxes instead of bubbles. We go through the same questions as in the example with seventh graders, just more slowly.

Even if students have experience in creating line graphs from data, we use these problems to build an understanding of the graph, the axes, and coordinates. We help the students plot the first solution from the first row of the table in **figure 1**: 5 Hershey's bars and 0 Tootsie Rolls. It is a definite step forward in abstraction for the students to record this solution as (5, 0) and plot that point in the graph in **figure 3**. We want to build students' *full* understanding of each representation and the connections among them. Therefore, we require that the students go back and connect to the real situation. We ask, "In the solution (4, 2), what does the 4 refer to?"

Sometimes we help students by identifying other points on the line and asking about a specific point. Is it a solution? All other points on this line would be noninteger "solutions," which are not permissible and are thus not solutions in this case. For instance, the six points in **figure 3** are the only solutions. They are on a straight line. "Why? What was the pattern in the table?" As with the seventh-grade example, we have students start at a solution point on their graph and move according to the pattern in the table (down by 1, up by 2). Some students need help seeing why moving to the left 1 point on the x-axis is going "down" (decreasing in value). Then they go up by 2 on the y-axis and arrive at the next point on the graph (4, 2). They continue and are excited to see that the pattern that they saw in the table has a direct counterpart in the graph.

Going Further with More Experienced Students

OUR MORE EXPERIENCED STUDENTS HAVE already worked with tables and graphs, and they quickly create data tables for the three problems we cited earlier ($10 to buy $2 and $1 items, $27 to buy $5 and $1 items, and $37 to buy $5 and $2 items). With good questions from the teacher, they can readily discern the patterns. We follow up with extension problems that give more money to spend on the same items as the first two problems (e.g., $15 to buy $2 and $1 items and $30 to buy $5 and $1 items). They begin to see that more money does not affect the pattern (e.g., respectively, down by 1, up by 2; or down by 1, up by 5).

Then we ask them to graph solutions to these ta-

bles. As in the seventh-grade example, we help them see that the pattern in the table has a corresponding visual pattern in the graph. They also see that when they make graphs of the same items but with more money, the solution points of the two tables will be in parallel lines, because they have the same "up by, down by" pattern. More money moves the line of solutions to the right. A few students notice that in some problems there will be solution points that lie on either one or both of the axes. We ask them, "Why? What does this mean?" Some students see right away that it may be possible to spend all the money on only one item if the total you have to spend is *divisible* by the cost of that item.

We make explicit that the points that lie on the axes are called *x-intercept* and *y-intercept*. We ask what the intercepts mean in this context. Having completed the table in **figure 2** and the graph of \$37 to buy \$5 and \$2 items, we tell them, "Think carefully about buying \$5 and \$2 items with \$40 to spend, but do not make a table yet. Instead, *predict* what the graph will look like." Some students are able to analyze the numbers and *reason* that with \$40, the solutions would include two intercepts—(8, 0) and (0, 20)—and that the solutions would be a line of points parallel to the \$37 solutions. We ask them to make the graph by drawing the axes on the graph paper; entering the x-intercept (8, 0); and using the pattern of down by 2, up by 5 to find all other solution points. Finally, they create the table to check their solutions.

We then formally introduce the concept of slope as a way of measuring the rate of change of these two variables. The students have developed a strong grounding of the changes in x and the corresponding changes in y through highly connected real-world examples, data tables, and graphs. We compare the change in y to the change in x and ask them to express this relationship in the standard convention $\Delta y/\Delta x$, which in our example would be $+5/-2$.

Students create equations by taking a table and its graph for which they now have a strong understanding (e.g., \$40 to buy \$5 and \$2 items). We proceed in the same manner as in the seventh-grade example. They generate $5x + 2y = 40$. But we do not stop there; for these students, we ask them to change the equation into the slope-intercept form, $y = mx + b$. When they manipulate the symbols, they get $y = (-5/+2)x + 20$. The students can see that the 5 and 2 come from the costs of the two items, +20 is the y-intercept, and the slope is negative because the pattern involves one variable increasing while the other decreases. Students who are working through these problems to review linear equations often say things such as "Now I understand. Why doesn't the book do it this way?"

Issues and Extensions

ONCE STUDENTS HAVE COMPLETED THE TASKS described above, understanding systems of linear equations becomes relatively easy for them. When they initially discuss the problem, they need to realize that two different sets of information are present. They can make two tables *or* make one table and use the second set of information as a constraint on the possible values of the variables in their single table. For instance, if students are told that they have \$40 to buy \$5 and \$2 items, ask them to consider that they must buy chocolate for eleven people. They could evaluate the solutions in their table to determine if any of the solutions in the single table have exactly eleven items being bought. The other option would be to create another table with solutions such as (11, 0); (10, 1); (9, 2); . . . ; (0, 11). They could graph these solutions on the same graph as $5x + 2y = 40$. The two tables would reveal only one common solution: (6, 5). Using the second table, they create the equation $x + y = 11$, which when graphed shows points on a "line" also with negative slope, intersecting the line of $5x + 2y = 40$ at point (6, 5).

The numbers chosen can vary greatly and determine the level of difficulty of a problem, the particular patterns, and the number of solutions. This versatility has enabled us to create problems that differentiate instruction for a diverse group of students. We can present scenarios to students, using devices, props, objects, pictures, or posters, to trigger their imaginations. Over the years, we have used many items that appeal to students, including types of apparel; audio/video/entertainment/cinema purchases; and snack foods. For instance, we have given students made-up gift certificates worth \$200 to Academy Awards Video Store with which to purchase DVDs at \$25 each and CDs at \$15 each. They must spend the entire value of the gift certificate. Students also enjoy creating their own problems, as did one younger student who wrote, "I want to buy Persian Vases that cost \$10,000 and Persian Rugs that cost \$4,000. I have a million dollars to invest. How can I invest all my money?" She had found six solutions, and the teacher challenged her and the class to find more. They found a lot more!

A bit more difficult are problems that involve two-digit decimals and monetary amounts using both dollars and cents. For example, students can be asked: If two snacks cost \$1.75 and \$1.25 each, determine all the possible solutions for spending \$35.00, with no change. The more experienced students had learned to check if the total was divisible by either cost. They set up a table and determined the solutions (20, 0) and (0, 28). **Figure 4** shows how the table reveals the upper and lower limits of

the two variables. We had introduced interpolation, so students used numbers that went halfway between 20 and 0 (the \$1.75 item) and between 0 and 28 (the \$1.25 item) to find the solution (10, 14). They interpolated twice more and found solutions of (5, 21) and (15, 7). Then they realized that the pattern was "down by 5, up by 7." By analyzing these results, some saw that the greatest common factor of a, b, and c was 0.25. They created the equation $1.75x + 1.25y = 35$ and changed it to $7x + 5y = 140$. Other students first changed it to the slope-intercept form, $y = -(1.75/1.25)x + (35/1.25)$, then simplified that expression to $y = (-7/5)x + 28$. They saw that the "28" came from the y-intercept (0, 28) and that the slope "–7/5" was a combination of two relatively prime numbers that make the pattern. On one occasion, a sixth grader found all the solutions quickly without doing any of these calculations. When asked, "How did you do this?" she replied, "I saw the down by 5, up 7 pattern." She was then asked, "How did you know that was the pattern?" She replied, "I just thought about what the two things cost: 7 quarters and 5 quarters."

The real-life contexts help students to reason effectively. They *think*; they do not abandon their mathematical reasoning and randomly select memorized but dimly understood procedures. The context invites them to imagine, reason, and create multiple representations. Our job as teachers is to issue an invitation in a way that challenges students but does not overwhelm them. We encourage them to make connections among their representations and build their understanding of the big ideas of algebra.

References

Burton, David M. *The History of Mathematics: An Introduction*. 2nd ed. Dubuque, IA: Wm. C. Brown, Publishers, 1991.

Lesh, Richard, Frank Lester, and M. Hjalmarson, "A Models and Modeling Perspective on Metacognitive Functioning in Everyday Situations Where Problem Solvers Develop Mathematical Constructs." In *Beyond Constructivism: Models and Modeling Perspectives on Mathematical Problem Solving, Learning, and Teaching,* edited by Richard Lesh and Helen M. Doerr. Mahwah, NJ: Lawrence Erlbaum Associates, 2003.

National Council of Teachers of Mathematics (NCTM). *Principles and Standards for School Mathematics.* Reston, VA: NCTM, 2000.

Robbins, R. *Beginning Number Theory.* Dubuque, IA: Wm. C. Brown, Publishers, 1993. □