SPOTLIGHT ON THE STANDARDS

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Representation as a Vehicle for Solving and Communicating

Stay Tuned: The "Spotlight" Has a New Focus in October

This article on representation brings to a close the set of ten articles that the *MTMS* Editorial Panel planned for this department. Under the expert guidance of department editors Carol Malloy and Diana Lambdin, "Spotlight on the Standards" has focused on the grades 6–8 Content and Process Standards found in NCTM's *Principles and Standards for School Mathematics* (2000).

The spotlight is still burning brightly, however, and the October issue will focus its beam on the principles, as envisioned in *Principles and Standards for School Mathematics.* The format will be similar to that of the current department, with the goal of providing an in-depth discussion of each of the six principles in turn. Examples of student work and vignettes of the principles in action will suggest topics for reflection and discussion with colleagues. The Editorial Panel invites you to respond to the topics raised in the new department through letters to the "Readers Write" section of the journal or with submissions to the editor of the new "Spotlight on the Principles," Jennifer Bay-Williams.

STUDENTS IN MS. SIMPSON'S SEVENTH-GRADE prealgebra class were challenged to use data to decide which of several class party plans was best. Because Takisha focused on price, she preferred using a table to justify her decision. Samantha used a written explanation to determine for herself which plan was best. Brandon's group used a table that indicated the price per person. However, when it came to convincing others of the best plan, some students chose other representations. For example, Brandon decided to use his graphing calculator to display linear graphs of the three plans. Melissa decided to use a triple bar graph. Tables, written explanations, rules, equations, and graphs were all important representations used to solve and communicate the results of the problem.

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What Are Representations?

REPRESENTATIONS ARE TOOLS THAT ARE VITAL for recording, analyzing, solving, and communicating mathematical data, problems, and ideas. Representation can be thought of as the language of mathematics (Coulombe and Berenson 2001). "The term *representation* refers both to process and to product—in other words, to the act of capturing a mathematical concept or relationship in some form and to the form itself" (NCTM 2000, p. 67). Representations allow us to experience abstract notions, such as 3/4, with physical materials or drawings. They allow us to provide a single number (e.g., median) to someone wanting to know the typical price of a new house. They allow us to powerfully model the growth of a population with a function that can then be efficiently manipulated. And they allow us to communicate our results to the world through graphs that speak volumes. Representations are important in that they are vehicles for learning and communicating; they support learners of many different styles; and they come in many different forms, allowing students to use combinations of representations to gain more information than would be possible with a single representation (Friedlander and Tabach 2001).

Representation as an NCTM Standard

RECENTLY, REPRESENTATION HAS COME TO THE forefront as a separate standard in *Principles and Standards for School Mathematics* (NCTM 2000). However, even though no standard was labeled "Representation" in *Curriculum and Evaluation Standards for School Mathematics* (NCTM 1989), a discussion of representations was integrated freely within the other grade 5–8 standards. For example, the Mathematical Connections Standard (in 1989) advocated a curriculum that would encourage students to "explore problems and describe results using graphical, numerical, physical, algebraic, and verbal mathematical models or representations" (p. 84). In addition to other examples from the Process Standards, representations were also featured in the Content Standards. The Algebra Standard maintained that "the mathematics curriculum should include explorations of algebraic concepts and processes so that students can represent situations and number patterns with tables, graphs, verbal rules, and equations and explore the interrelationships of these representations" (NCTM 1989, p. 102).

The authors of *Principles and Standards* chose to feature representations more prominently by providing a separate standard. The following is the statement of the Representation Standard:

> Instructional programs from prekindergarten through grade 12 should enable all students to—
>
> - create and use representations to organize, record, and communicate mathematical ideas;
> - select, apply, and translate among mathematical representations to solve problems;
> - use representations to model and interpret physical, social, and mathematical phenomena.

Note that a lot of action is depicted in this Standard. In particular, students should be able to use representations that they are given, select from various representations those that are useful for a specific situation, and create their own representation. At various times, the goal of a mathematical activity might be for students to understand and use a given representation, to provide a representation, choose a representation as one step in solving a problem, and use a representation to communicate the results of a problem or to assist someone else in making sense of a situation. The Standard also values the ability to make connections between representations. Additionally, representations can be used to effectively model a wide range of real-world phenomena.

The Class Party Problem

Ms. Simpson formed a committee of students to investigate sites for a class party. The committee does not know how many of the students will attend and has not yet decided whether each class member can invite a guest. However, they have brought back the following details on parties:

- Water World ☛ Swimming, hot dog, chips, and drink. Cost: $100 to reserve the pool and $5 per person.
- Pizza Pi and MoviePlex ☛ Pizza, drink, movie. Cost: $10 per person.
- Skate 'Til Late ☛ Skating and ice cream (eat before you come). Cost: $200 to rent the skating rink and $2 per person for skate rental.

If price is the primary focus of your decision, which party option is the best?

How would you convince other class members of this?

How would you advise other groups planning a party, so that they could make a decision, no matter how many guests they had?

Fig. 1 The Class Party problem

people	waterw.	Pizza Pi	Skate "Till late)
100	$600	$1000	$460
125	$725	$1250	$535
150	$850	$1500	$610
175	$975	$1750	$685
10	$150	$100	$250
20	$200	$200	$250
30	$250	$300	$250
40	$300	$400	$280
120	$700	$1200	$520
130	$750	$1300	$550
140	$800	$1400	$580

Fig. 2 Melissa's table

Using Representations to Solve and Communicate Results from the Class Party Problem

IN THIS ACTIVITY, WE CHOSE TO FOCUS ON THE use of representations to solve a problem and communicate the results. Ms. Simpson instructed the students to "use whatever mathematics you can think of" to solve the Class Party problem (see **fig. 1**).

Students worked in groups of three to five and were reminded that they could use rulers, graph paper, calculators, algebra tiles, and other tools. As the students began to work, we circulated around the room, making sure students understood the task. Initially, a number of students sought "one right answer." A few students wanted to know exactly how many students would be attending the party. A couple of students wanted to select a party based on personal preference instead of finances.

Several students began choosing numbers of party-goers and determining the best plan for those specific numbers. For example, one group focused on 60, believing that this number would cover all class members, one friend each, and sponsors. When other groups began to see that the best plan changed as the number of people attending changed, they started to organize tables. Some students proceeded in rather random fashion when creating their tables, but others took a more organized approach and found values for every five or ten students. Tables emerged as the leading representation in terms of helping students decide for themselves which of the three plans was best for a particular number of party-goers. **Figure 2** is an example of one such table. Note the rather scattered, yet somewhat systematic, approach.

Representations Used

DURING THE COURSE OF THE ACTIVITY, WE OBserved the following representations:

- Word rules (e.g., Pizza Pi and Movie Plex—less than 20 people is cheapest)
- Informal and formal language (e.g., steepness, slope, rate)
- Tables (including groups using a single table and one group using a series of tables)
- Graphs—line (both hand and calculator generated) and bar (triple bar graph for each of several numbers of people)

- Equations (e.g., $y = 5x + 100$ for Water World)
- Patterns—some students noticed that for every 5 new people, one plan increased \$25; another, \$50; and another, \$10. Plans that started out expensive became more reasonable.
- Hand gestures (e.g., motioning how one plan was catching up to another in terms of expense; demonstrating slope or rate)

Discussion

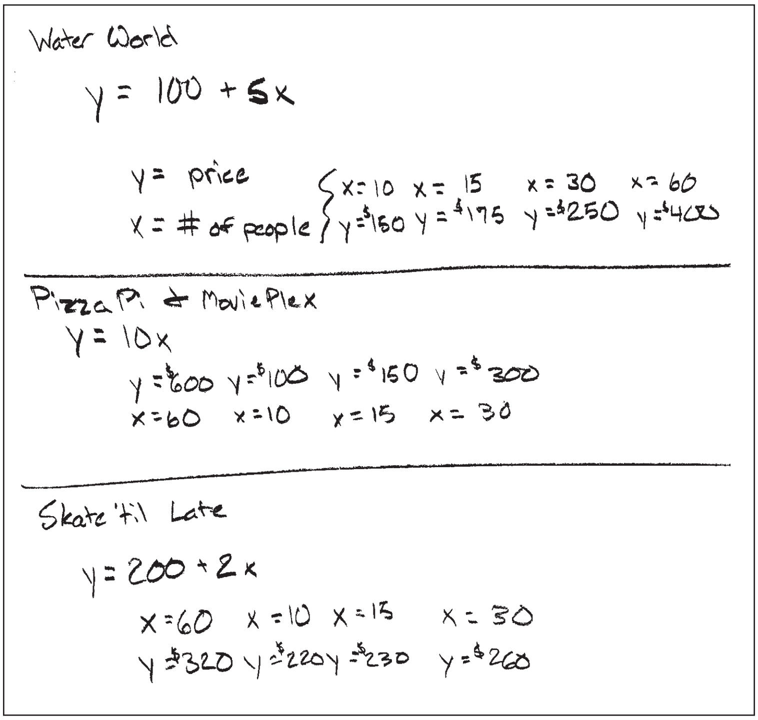
GIVEN THE NATURE OF THE TASK, WE WERE NOT too surprised that some common representations—drawings and concrete materials—were not used. The numbers (\$100 and \$200 rental fees) were not conducive to algebra tiles, and drawings were not necessary for organizing information from the problem. (Some students did use drawings in peripheral ways; for example, drawing skates to decorate their table.)

Due to the context, certain representations were more helpful for the students than others.

- Tables provided a good organizational structure for the initial guess-and-check approach.
- Graphs provided useful visuals for groups wishing to communicate their results.
- Equations provided a compact statement that summarized the work of the students.

After students read the party options, they began to discuss which package would be the best plan. Students developed explanations for why a particular package would be the best to choose, some of them nonmathematical. For example, one student stated, ". . . water is probably most fun and it holds a reasonable amount of people. When you are watching a movie, you can't talk." When we asked students for evidence supporting a particular choice, they began to think about mathematical ways to represent their answers.

Most students used tables in the beginning of their problem solving. The tables varied in appearance, but the majority of students used a four-column table that included the number of people and a column for the cost of each party package. One group created three separate tables in which there were three columns. The first column represented the number of people; the second, the price per person; and the last, the total cost. This group was able to use the information to support its answer; however, group members ultimately turned to equations to provide a general solution. (See this group's algebraic thinking in **fig. 3.**) Tables that included number of people and costs worked well for students who transferred their data to another representation—graphing.



Water World

$y = 100 + 5x$

y = price
x = # of people

$x = 10$, $y = \$150$; $x = 15$, $y = \$175$; $x = 30$, $y = \$250$; $x = 60$, $y = \$400$

Pizza Pi + MoviePlex

$y = 10x$

$y = \$600$ $y = \$100$ $y = \$150$ $y = \$300$
$x = 60$ $x = 10$ $x = 15$ $x = 30$

Skate 'til Late

$y = 200 + 2x$

$x = 60$ $x = 10$ $x = 15$ $x = 30$
$y = \$320$ $y = \$220$ $y = \$230$ $y = \$260$

Fig. 3 One group used equations to find the best plan.

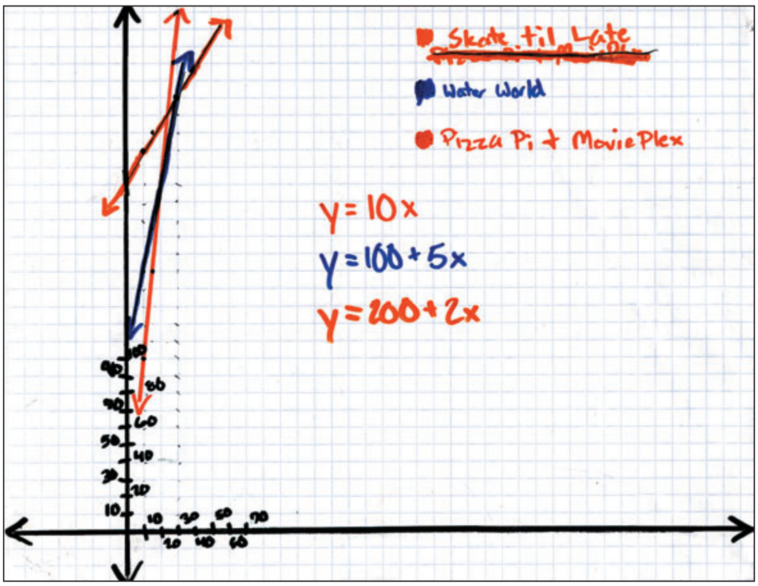


Fig. 4 Traci's graph

Groups used several styles of graphs—line, bar, and pie (we omit discussion of the pie graph here since it was not very useful). Students learned which style of graph worked best for this situation and why particular graphs are used for certain problem situations. The students who attempted line graphs typically labeled the horizontal axis with the number of people and the vertical axis with the cost. Then they placed points on the grid and connected them. **Figure 4** shows a sample graph. Note that the scaling is not optimal (a common difficulty) and that this group also eventually summarized their thinking with a series of equations.

REPRESENTATION	TYPICAL USES	ADVANTAGES	DISADVANTAGES
Verbal	Presenting the original problem, communicating with others while solving the problem, and reporting the final results	Uses students' natural language and often helps to connect the problem to the real world	Natural language can be ambiguous, particularly as compared with precise mathematical language
Pictorial	Gathering information from the problem; modeling movements in an action problem	Helps students "see" the mathematical situation; comfortable approach for most middle grades students	Students sometimes make assumptions with drawings that go far beyond the problem statement (e.g., triangle drawn as equilateral); some students' drawing skills are poor
Numerical/ tabular	Early work used in understanding a problem; finding specific examples that fit the context; guessing and checking; often organized in tabular form	Natural tool for most middle-grades students based on previous experience; can serve as an effective tool for bridging to both graphs and equations	Lack of generality can impede progress; use of only certain numbers (e.g., whole numbers or multiples of 10) may obscure key situations
Graphical	Useful for showing increases, decreases, maximums, minimums, intersections; particularly useful for communicating results	Demonstrates clear pictures of trends; is intuitive for most students; appeals to visual learners	Scaling and accuracy issues often lead to misleading graphs; students often draw continuous graphs for discrete data (which is not always bad)
Algebraic	When students begin to feel confident enough about a context to generalize; a minority of students gravitate naturally to this representation	Provides a concise and general statement of a situation; once in place can be manipulated easily to provide results, including those involving, for example, fractions; useful for justifying	May not communicate meaning to other students; is most difficult for most students initially, based on lack of previous experience; early reliance may obscure meaning that tables and graphs provide

Fig. 5 Uses, advantages, and disadvantages of representations

One group used a bar graph with a legend in which each party package was distinguished by a color. The group had the horizontal axis labeled "number of people" and the vertical axis labeled "cost." This graph was an interesting way to represent and support their findings; however, it was difficult for the group to discuss varied scenarios because they were limited to the data shown in their graph.

Equations began to appear near the end of the activity after students analyzed their work and discussed it among themselves and with the instructors. Students were at first hesitant to express their work with equations. However, after reviewing the other representations (tables, word rules, graphs), a few groups found that the equations helped produce a very useful summary.

Toward the end of the session, students seemed to better understand the connections between the varieties of representations offered. Initially, they wanted to simply talk about their choices and provide support for their own work. After some discussion, students began to explicitly express connections and offer their opinions as to the best representations for this activity.

Given the different uses of representations, we decided to summarize how we see middle school students using them and what we see as the advantages and disadvantages of each (see **fig. 5**). We adapted our table from Friedlander and Tabach (2001).

Lessons Learned

WHEN STUDENTS ARE ASKED TO choose their own representation, they sometimes choose their favorite or the representation with which they are most comfortable, not what is most useful. For example, we had students try to use a pie graph to assist in making decisions about the best party plan. Although a pie graph is a visual representation that these students had come to appreciate from past experience, it did not help them organize or communicate their thinking in this context.

Allowing students to choose their own representations may not allow the teacher to focus on a particular form. Working on a specific representation can be done in at least a couple of ways: (1) introduce the problem using that representation or (2) require that students use a particular representation for their solution (see Coulombe and Berenson [2001]). However, we valued allowing students to choose their own representations on this occasion.

Students typically used different representations for communicating to other students than they did to solve the problem. We found it interesting that many students used numerical approaches for their work and then when they were satisfied with the solution, turned to graphs to convince others. The positive aspect of this action was that sometimes the graphical representation gave students additional information for the problem that they believed they had already solved.

Students found tables to be very useful, but often gaps appeared in what they could tell us. For example, since Skate 'Til Late does not become the most economical plan until the thirty-fourth person attends the party, a group looking only at multiples of 10 for their numbers of people might not believe this change occurs until the fortieth person.

As noted, many students believed that graphs had great potential for convincing others of their conclusions. However, they were not always able to devise the proper scaling to produce the picture that made their case. Although we did not discuss it at that time, we were somewhat concerned about using a continuous representation to model a discrete situation. We are not necessarily opposed to doing that, since the power of mathematical representations sometimes resides in their ability to be "stretched" for use in additional contexts. However, we wondered if the time was appropriate to hold a discussion about continuous and discrete data. In the end, we chose not to have this discussion.

Final Thoughts

TABLES, GRAPHS, EQUATIONS, AND word rules are powerful representations for middle grades students. Each appears to have its own place in the student reasoning process that underlies his or her choice of solution strategy and the choice of communication tool. This situation alone suggests the importance of teachers exposing students to multiple representations so that they will have choices when solving or communicating.

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